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J. Phys. A: Math. Gen. **33** (2000) 3771–3782. Printed in the UK PII: S0305-4470(00)08890-9

Transfer formalism for quantum optics problems

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Received 20 October 1999

Abstract. A consistent quantum formalism based on the localized basis of the Wannier functions in the Heisenberg and Schrödinger pictures to describe propagation of an electromagnetic field in a three-dimensional media including diffraction is presented. In the Schrödinger picture the Fokker–Planck equation for the Glauber–Sudarshan quasi-probability and the corresponding Langevin equations are given. As a result the spacetime description is obtained by a simple change of variables in the temporal master equation of the field. Using this formalism it is shown that the existence of integrals of motion in the propagation of light in a medium under the condition of non-degenerated parametric and two-photon interactions results in amplification of modes when non-classical properties of the light are conserved. Quantum propagation of light in a linear medium taking into account the diffraction is considered and its solution is found.

1. Introduction

When considering the statistics of light the usual approaches are based on the temporal evolution of the electromagnetic field. It is natural for problems when the spatial behaviour can be neglected, such as for the high-Q optical cavity scheme for which the temporal features are important. In contrast, the spatial behaviour plays the key role in propagation of light through media or a distributed system that cannot be considered as a point. In these systems new states of a light called spatially squeezed states arise [1]. The quantum formalism for distributed systems has been developed in a number of works. One-dimensional approaches were presented in [2]. A theory for continuous variables in the Heisenberg picture was given in [3], where the continual coherent states and squeezed states of the field have been introduced and a theory of light detection in free space was presented. Using these approaches, evolution of the light statistics has been examined for the three- [4] and four-photon [5] parametric interaction in transparent media. Multi-photon processes were discussed in [6].

In this paper we present a quantum formalism, which enables us to consider the propagation of light in a three-dimensional medium with diffraction being taken into account. It is based on a localized basis of the Wannier functions, and allows the conversion of the temporal description to a spacetime description by a simple change of variables. Indeed, it is the method used in solid-state physics for the transition from collective to local variables [7]. The main feature of the local description in the Wannier basis is that the problem turns out to be multi-particle when the local field oscillators interact among themselves even in free space. From the physical point of view this is a transfer of excitation or the light propagation process. On the other hand, the interaction of local oscillators produces the coupled equations, which as we show have the form of the transfer equations. Our approach is formulated in both the Heisenberg and

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Schrödinger pictures where the Fokker–Planck equations for the field quasi-probability are derived together with the corresponding Langevin equations.

To describe the interaction of light with atoms, the adiabatic illumination of fast atomic variables is often used to obtained a closed equation for the field or master equation [8, 9] that is a starting point for analysing the statistics of the light. The simple recipe of how to obtain the local description allows us to obtain the transfer equation immediately from the temporal master equation by missing all steps of the derivation procedure.

The potential of the presented formalism is illustrated using two problems. In the first problem the statistics of the light are considered for propagation in a one-dimensional medium with two-photon and parametric interaction. Here within the framework of a local description an interesting peculiarity due to the integrals of motion arises. The existence of the integrals makes it possible to establish the main features of the statistics transformation immediately without going into the solution of the dynamic problem [10]. It has been shown in this way that enhancement of light and the conservation of non-classical properties can be possible. In the second problem the propagation of the light in a linear three-dimensional medium with diffraction is considered and solutions have been found.

2. Wannier basis. Local operators

In one-dimensional normalization space L plane waves $\varphi_k(x) = (1/\sqrt{L})\exp(\mathrm{i}kx)$, where $\Delta k = 2\pi/L$, form a complete orthonormal basis. It is used for a standard representation of the electromagnetic field strength operator, where the operators of photon creation and annihilation $a_k^\dagger$, a_k arise with commutational relations

$$[a_k, a_{k'}^\dagger] = \delta_{kk'}. \tag{2.1}$$

The operators $a_k^\dagger$, a_k describe the creation and annihilation of photons of a wavevector k over the whole space L. These operators may be called collective, because they are responsible for the excitation of the whole volume. For the local description of the electromagnetic field instead of plane waves we use Wannier functions known in solid-state physics [7], which are packets of plane waves

$$w_m(x-l) = \frac{1}{\sqrt{N}}\sum_{k\sim m}\exp(-\mathrm{i}kl)\,\varphi_k(x) \tag{2.2}$$

the wavevectors $k \sim m$ lie in a band $m - \pi/a \leqslant k < m + \pi/a$. Here each band m, or zone, corresponds to the partition of one-dimensional space L into $N = L/a$ cells, with positions determined as $l = na$. In k-space the centres of so-defined bands are separated by an interval $\Delta m = 2\pi/a$. The terms in (2.2) form a geometrical progression, so summation is possible. Then it easily verified that for large N the Wannier functions are localized in a cell with a coordinate l in the region $\sim a$. The Wannier functions defined in accordance with (2.2) form the complete orthonormal set

$$\sum_{ml} w_m(x-l)\,w_m^*(x'-l) = \delta(x-x') \tag{2.3}$$

$$(w_m(x-l), w_{m'}^*(x-l')) = \delta_{mm'}\delta_{ll'} \tag{2.4}$$

where the scalar product is defined by the integration over the whole space L.

The Wannier basis serves as a framework for the introduction of local operators of creation and annihilation $a_{ml}^\dagger$, $a_m(l)$:

$$\sum_k \varphi_k(x)a_k = \sum_{ml} a_m(l)\,w_m(x-l). \tag{2.5}$$

From (2.5) it follows that the operators are coupled by a unitary transformation

$$a_m(l) = \sum_k C^*_{mk}(l)\, a_k \tag{2.6}$$

$$a_k = \sum_{ml} C_{mk}(l)\, a_m(l) \tag{2.7}$$

where

$$C_{mk}(l) = \frac{1}{\sqrt{N}} \sum_{k' \sim m} \exp(-\mathrm{i}k'l)\, \delta_{k'k}. \tag{2.8}$$

For local operators the following commutational relations are valid:

$$[a_m(l), a^\dagger_{m'}(l')] = \delta_{mm'}\delta_{ll'}. \tag{2.9}$$

Relations (2.5) allow us to interpret them as operators of creation and annihilation of a photon at a point l in the vicinity $\sim a$. As a result local field oscillators described by the introduced operators are defined in the space L. From here on we will refer to the packets of plane waves forming the local operators as local modes with wavenumber m and with a width $\Delta\nu_m = c/a$. Strictly speaking these packets are not modes, because there is no physical reason for them to be distinguishable, however, the term seems to be convenient.

Equation (2.5) is a starting point in the formalism of quantum transfer theory, which may be formulated in different pictures. From the operational point of view the unitary transformations (2.6) and (2.7) following from (2.5) mean that the transition from the non-local to the local description and vice versa is accomplished by a change of variables.

The specific feature of the local description is its many-particle character, where the local oscillators are already interacting in the free space. This interaction describes the excitation transfer or light propagation process, which determines the structure of the equations of motion, where chains of BBGKI type (Bogolyubov, Born, Green, Kirkwood, Ivon) arise, resulting finally in the propagation equations.

Consider the case of free space, for which the field evolution is defined only by a Hamiltonian $H_0 = \sum_k \hbar\omega_k a^\dagger_k a_k$, $\omega_k = ck$. Let the operator a_k in (2.5) be defined in the Heisenberg representation, i.e. satisfying the equation $\partial a_k/\partial t = \mathrm{i}\hbar^{-1}[H_0, a_k]$. To find the equation for the local Heisenberg operator

$$a_m(l,t) = \frac{1}{\sqrt{N}} \sum_{k \sim m} a_k(t) \exp(\mathrm{i}kl) \tag{2.10}$$

let us differentiate with respect to time. Assuming the size of the space cell a to be small, $a \ll L$, and $a \to 0$, l may be considered as a continuous space coordinate. In this approximation

$$k \exp(\mathrm{i}kl) = -\mathrm{i}\frac{\partial}{\partial l} \exp(\mathrm{i}kl) \tag{2.11}$$

as a result it follows that the transfer equation for the local operator is

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial l}\right) a_m(l,t) = 0. \tag{2.12}$$

This equation is valid over a coarse space scale with a characteristic size a providing the fulfilment of (2.11), in this case the replacement $\delta_{ll'} \to a\delta(l - l')$ is possible.

For a many-particle problem (2.12) may be represented as a set of coupled equations due to the interaction of local operators. The Hamiltonian H_0 appears to be non-diagonal with respect to the indices ll'. The equation of motion takes the form

$$\frac{\partial}{\partial t}a_m(l) = \mathrm{i}\hbar^{-1}[H_0; a_m(l)] = -\mathrm{i}\sum_{l'}\Omega_m(l,l')\,a_m(l') \tag{2.13}$$

the coupling constant

$$\Omega_m(l,l') = \frac{c}{N}\sum_{k\sim m} k\exp(\mathrm{i}k(l-l')) \tag{2.14}$$

links the oscillator at the point l with all of its neighbours. However, due to its δ-shape, the coupling appears to be significant only for two adjacent oscillators. In other words, in free space the interaction between local oscillators is described on a coarse scale by a derivative with respect to l:

$$\frac{\partial}{\partial l}a_m(l) = \frac{\mathrm{i}}{c}\sum_{l'}\Omega_m(l,l')\,a_m(l'). \tag{2.15}$$

Equation (2.15) is obtained by differentiation of (2.10) with respect to l, taking (2.11) into account. Consider the commutator of local operators for different times. For the free evolution, when the dynamics of operators is determined by the Hamiltonian H_0

$$[a_m(l,t); a^{\dagger}_{m'}(l,t+\tau)] = \delta_{mm'}\exp(\mathrm{i}cm\tau)\frac{1}{\Delta\nu_m}\delta_a(\tau) \tag{2.16}$$

the function (2.17)

$$\delta_a(\tau) = \frac{\Delta\nu_m}{N}\frac{\sin(\pi\tau\Delta\nu_m)}{\sin(\pi\tau\Delta\nu_m N^{-1})} \tag{2.17}$$

$N \gg 1$ has a sharp maximum for $\tau \to 0$. Its weight is concentrated in the vicinity of the order of $a/c = \Delta\nu_m^{-1}$, so it may be considered as a delta-function when $\Delta t \geqslant a/c$. Note that $\delta_a(0) = \Delta\nu_m$. The occurrence of the temporal scale delta-function is connected with the use of a coarse space scale with characteristic size a, where the time interval will be $\Delta t \geqslant a/c$.

Let us introduce the interaction picture. Consider the slowly changing part or envelope of the local operator $a_m(l)$:

$$a_m(l,t) = A_m(l,t)\exp(-\mathrm{i}\omega_m t + \mathrm{i}ml) \tag{2.18}$$

where $\omega_m = cm$. Then the unitary transformations in (2.6) and (2.7) will take the form

$$A_m(l,t) = \frac{1}{\sqrt{N}}\sum_{k\sim m} a_k(t)\exp\{-\mathrm{i}(\omega_k-\omega_m)t + \mathrm{i}(k-m)l\} \tag{2.19}$$

$$a_k(t) = \frac{1}{\sqrt{N}}\sum_{l} A_m(l,t)\exp\{\mathrm{i}(\omega_k-\omega_m)t - \mathrm{i}(k-m)l\} \tag{2.20}$$

the evolution of the operators a_k and $A_m(l,t)$ is defined only by the Hamiltonian of interaction V,

$$\frac{\partial}{\partial t}a_k(t) = \mathrm{i}\hbar^{-1}[V(t), a_k(t)] \tag{2.21}$$

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial l}\right)A_m(l,t) = \mathrm{i}\hbar^{-1}[V(t), A_m(l,t)]. \tag{2.22}$$

3. Three-dimensional case. Quasi-optical approximation

Consider a normalized volume L^3 with a number of cells $N = N_1N_2N_3, N_1 = N_2 = N_3 = l/a$, where for simplicity the cell was chosen to be of cubic shape. Then in the initial formulae the evident replacements will be $k \to \boldsymbol{k}$, $m \to \boldsymbol{m}, l \to \boldsymbol{l}$, $x \to \boldsymbol{r}$. For the local operator in the interaction picture the following expression should be written instead of (2.19):

$$\boldsymbol{A}_m(\boldsymbol{l}, t) = \frac{1}{\sqrt{N}} \sum_{k \sim m} \boldsymbol{a}_k(t) \exp(-\mathrm{i}(\omega_k - \omega_m)t + (\boldsymbol{k} - \boldsymbol{m})\boldsymbol{l}) \tag{3.1}$$

where the dispersion law has the form $\omega_q = c\sqrt{(\boldsymbol{q}, \boldsymbol{q})}$, $\boldsymbol{q} = \boldsymbol{k} \cdot \boldsymbol{m}$. Let us find the equation of motion for local operators. Similarly to the one-dimensional case let us differentiate (3.1) with respect to time. Assuming $\boldsymbol{l}$ to be a continuously changing variable, we will find instead of (2.11)

$$\sqrt{(\boldsymbol{q}, \boldsymbol{q})} \exp(\mathrm{i}\boldsymbol{q}\boldsymbol{l}) \approx \left(-\mathrm{i}\frac{\partial}{\partial l_z} - \frac{1}{2q_z}\left(\frac{\partial^2}{\partial l_x^2} + \frac{\partial^2}{\partial l_y^2} \right) \right) \exp(\mathrm{i}\boldsymbol{q}\boldsymbol{l}). \tag{3.2}$$

When summing over $\boldsymbol{k} \sim \boldsymbol{m}$ we replace $k_z \to m \approx m_z$. This approximation corresponds to a quasi-plane wave of frequency $\omega_m = cm$, propagating along the z-axis. In as much as a coarse spatial scale was introduced, for which vector $\boldsymbol{l}$ may be considered as a continuous function of the coordinates, let us make a replacement $\boldsymbol{l} \to \boldsymbol{r}(x, y, z)$. As a result the equation of motion takes the form

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial z} - \mathrm{i}\frac{c^2}{2\omega_m}\left(\frac{\partial^2}{\partial x} + \frac{\partial^2}{\partial y} \right) \right) \boldsymbol{A}_m(\boldsymbol{r}) = -\mathrm{i}\hbar^{-1}[V, \boldsymbol{A}_m(\boldsymbol{r})] \tag{3.3}$$

where V is the Hamiltonian of the interaction. The equation following from (3.3) for the free field ($V = 0$) is well known in the classical theory. It describes light propagation with diffraction taken into account in a quasi-optical approximation.

In equation (3.3) the Hamiltonian V describing the electromagnetic field interactions with the medium should be expressed in terms of local operators. As an example consider the Hamiltonian of light interaction with atoms in the dipole approximation, which is often a basis for a variety of problems. It has the form

$$V(t) = -\mathrm{i}\sum_{A,\boldsymbol{k}} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 L^3}}\boldsymbol{a}_k \exp(-\mathrm{i}\omega_k t + \mathrm{i}\boldsymbol{k}\boldsymbol{r}_A) d_A(t) + \mathrm{h.c.} \tag{3.4}$$

Equation (3.4) is given in the interaction picture, where d_A is the operator of the dipole moment for an atom located at a point $\boldsymbol{r}_A$. Changing to local operators with the aid of (2.20), where the three-dimensionality should be taken into account, we will use the following approximations. Let the packet or the local mode interact with the atom as a whole. It means that inside the band $\Delta\nu_m$ all frequencies $\omega_k \approx \omega_m$. Replace the atom position $\boldsymbol{r}_A$ by the position of the cell where this atom is located. Then the summation over the atoms may be divided into a sum over the cells $\boldsymbol{l}$ containing atoms and a sum over the atoms inside the cell. Suppose d_l is the operator of the atomic dipole moment in the cell $\boldsymbol{l}$, then equation (3.4) will take the form†

$$V(t) = -\mathrm{i}\sum_{m\boldsymbol{l}} \sqrt{\frac{\hbar\omega_m}{2\varepsilon_0 a^3}}\boldsymbol{A}_m(\boldsymbol{l}) \exp(-\mathrm{i}\omega_m t + \mathrm{i}\boldsymbol{m}\boldsymbol{l}) d_l + \mathrm{h.c.} \tag{3.5}$$

† Details of these and following calculations are given in [11].

Here a new normalization volume a^3 appears, while the summation is performed only over the cells containing atoms. The Hamiltonian obtained describes the elementary interactions of the local field oscillators or photons in cells l with atoms located inside. As a result changing to local oscillators in the interaction Hamiltonian (3.4) reduces to an ordinary replacement: $\boldsymbol{a}_k \to \boldsymbol{A}_m(\boldsymbol{l})$, $\boldsymbol{k} \to \boldsymbol{m}$, $\boldsymbol{r} \to \boldsymbol{l}$.

In a number of cases effective interaction operators obtained, for example, by unitary transformations of the starting Hamiltonian (3.4), are used for the description of multi-photon processes. Given below are two effective interaction operators for two-photon and parametric interactions in the presentation of local operators:

$$H_2 = \sum_{\boldsymbol{l}} \sum_{1,2} f_{12} \boldsymbol{A}_{m_1}(\boldsymbol{l}) \boldsymbol{A}_{m_2}(\boldsymbol{l}) S_l \exp(\mathrm{i}(\omega_{21} - \omega_{m_1} - \omega_{m_2})t) + \text{h.c.} \tag{3.6}$$

$$H_3 = \sum_{\boldsymbol{l}} \sum_{1,2,3} G_{123}(\boldsymbol{l}) \boldsymbol{A}_{m_1}(\boldsymbol{l}) \boldsymbol{A}_{m_2}(\boldsymbol{l}) \boldsymbol{A}^{\dagger}_{m_3}(\boldsymbol{l}) \exp(-\mathrm{i}(\omega_{m_1} + \omega_{m_2} - \omega_{m_3})t) + \text{h.c.} \tag{3.7}$$

where f_{12}, $G_{123} = g_{123} \exp(\mathrm{i}(\boldsymbol{m}_1 + \boldsymbol{m}_2 - \boldsymbol{m}_3)\boldsymbol{l})$ are the coupling constants. The Hamiltonian H_2 describes two-photon interaction of the modes having frequencies in the region of two-photon resonance: $\omega_{21} \approx \omega_{k_2} + \omega_{k_2}$. The operator $S_l = |2\rangle_l \langle 1|$ corresponds to the transition of atoms located at the point l from the lower to the upper working level. The process of parametric interaction of three waves in a transparent medium conforms to the Hamiltonian H_3.

4. Integral of motion and the statistics of the light

The integrals of motion may appear in the problems of light propagation in a medium, for which it is natural to use a local description. The existence of the integrals enables one to examine some peculiarities of the light statistics transformation without going into the solution of dynamic equations. Consider as an example two nonlinear processes: two-photon and parametric interactions, which are described by the effective Hamiltonians (3.6) and (3.7). Both processes are multimode, involving all pairs and triplets of modes having frequencies related as

$$\omega_{m_1} + \omega_{m_2} = \omega_{21} \tag{4.1}$$

$$\omega_{m_1} + \omega_{m_2} = \omega_{m_3}. \tag{4.2}$$

Hereafter we restrict ourselves to the case of a one-dimensional medium assuming the light to be propagating along the z-axis.

The pairs of modes m_1 and m_2 connected by conditions (4.1) and (4.2) and (4.31) will be referred to as conjugated. For them the difference of the photon number operators

$$I = n_1(l) - n_2(l) \tag{4.3}$$

where $n_j = A^{\dagger}_{m_j}(l) A_{m_j}(l)$, $j = 1, 2$ commutate with the Hamiltonian

$$[H_{2,3}, I] = 0. \tag{4.4}$$

Consequently, I and any function of the form $f(I)$ are integrals of motion. For a problem with boundary conditions l will be considered as a continuous coordinate replacing $l \to z$. Introducing the new variables and assuming the light propagating along the z-axis: $t' = t - z/c$, $z' = z$. Then by virtue of (4.4) the difference of the number of photons in conjugated modes is conserved for any point of the medium:

$$I(z, t) = I(0, t - z/c). \tag{4.5}$$

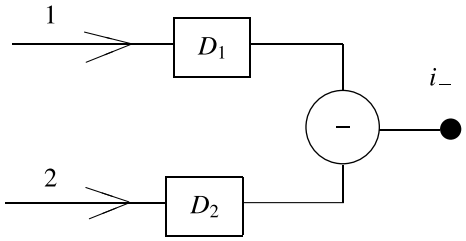

Figure 1.

With the use of the obtained operator integrals of motion it is possible to find the peculiarities of the transformation of the mutual correlation of conjugated modes for the propagation in a medium. Thus all statistical properties described by the correlation functions of different intensity, e.g. of the form $\langle (I(z,t))^p I((z,t+\tau))^q\rangle = K^{(p,q)}(z,\tau)$ are conserved by virtue of (4.5). From the standpoint of the observation, the lowest-order correlation function and its Fourier image is of more direct interest to us:

$$K(z,\Omega) = \int_{-\infty}^{\infty} \langle I(z,t) I(z,\tau)\rangle \exp(\mathrm{i}\Omega\tau)\,\mathrm{d}\tau. \tag{4.6}$$

It is possible to measure the function (4.6) in a scheme with two photodetectors (figure 1), where the fluctuation spectrum of the difference photocurrent $i^2(\Omega)$ is observed. For this scheme

$$i^2(\Omega) = \eta\langle n_1 + n_2\rangle + \eta^2 K_N(\Omega). \tag{4.7}$$

Here the first term which is independent of frequency is the shot noise. The index N indicates normal ordering of the field operators to describe a detector that responds to the absorption of a photon. The rate of registration $\eta = q\Delta\nu_m$, where q is the quantum efficiency of the detectors, which for simplicity are assumed to be equal. The presence of the local mode $\Delta\nu_m = c/a$ in the expression for the photocurrent spectrum follows from the formulae for detection given in the presentation of local operators. This may be elucidated as follows. A wide-band photodetector is needed to describe the light detection. Therefore, to register the local mode in front of the detector we have to place an optical filter of bandwidth $\Delta\nu_m$. Then $\Delta\nu_m$, determining the rate at which photons fall on the detector, is the bandwidth of the scheme.

Assuming that at the detector the local operators commute as free-field operators (2.16), let us to write the correlation function (4.6) in the normal and time-ordered form $K(\Omega) = K_N(\Omega) + \langle n_1 + n_2\rangle/\Delta\nu_m$. Then taking into account the motion integral one finds

$$K_N(z,\Omega) + \frac{1}{\Delta\nu_m}\langle n_1 + n_2\rangle = K_N(0,\Omega) + \frac{1}{\Delta\nu_m}\langle n_{10} + n_{20}\rangle \tag{4.8}$$

where $\langle n_{10}\rangle$ and $\langle n_{20}\rangle$ are the input photon number. As a result the expression for the photocurrent or noise spectrum (4.7) takes the form

$$i^2(z,\Omega) = \Delta\nu_m\langle n_1 + n_2\rangle q(1-q) + \Delta\nu_m\langle n_{10} + n_{20}\rangle q(q-1) + i^2(0,\Omega) \tag{4.9}$$

where

$$i^2(0,\Omega) = \Delta\nu_m q\langle n_{10} + n_{20}\rangle + (\Delta\nu_m q)^2 K_N(0,\Omega) \tag{4.10}$$

is the input noise.

It is seen from (4.9) that for an ideal photodetector ($q = 1$) the input and output noise spectra of the light are equal. This means that the correlation of conjugated modes is conserved during propagation in the medium. Suppose that the input state of the modes is non-classical, so it is a quantum correlation for which the shot noise is suppressed in bandwidth $\Delta\nu_m$ by a factor of $1 - q$:

$$i^2(0, \Delta\Omega) = \Delta\nu_m q \langle n_{10} + n_{20}\rangle(1 - q). \tag{4.11}$$

The state of the light resulting in (4.11) is produced, for example, by an optical parametric amplifier (OPA). Then the level of shot-noise suppression in the output is unchanged:

$$i^2(z, \Delta\Omega) = \Delta\nu_m q \langle n_1 + n_2\rangle(1 - q). \tag{4.12}$$

This means that the quantum properties of light in the medium are conserved. Moreover, in the medium with two-photon or parametric interaction the conjugated modes may be amplified, in that the observed initial correlation, in particular, the quantum correlation is conserved. This specific feature determines the properties of spontaneous radiation. Thus, if the input is the vacuum state, then $i^2(z, \Delta\Omega) = 0$ at the output. This means that the conjugated modes arising in spontaneous radiation have a non-classical correlation that results in a suppression of shot noise. Indeed, all these properties follow from the existence of integrals of motion in systems expanded in space.

5. Local quasi-probabilities

In previous items the Heisenberg picture was introduced for local operators. In the Schrödinger picture the evolution is determined by the density matrix of the electromagnetic field ρ, which may be connected to c-number functions $P(\{\alpha_k\}, s)$ called s-ordered quasi-probabilities. They arise in the density matrix expansion over operators $\Delta(\{\alpha_k\}, s)$, which are Fourier images of s-ordered displacement operators forming the complete set:

$$\Delta(\{\alpha_k\}, s) = \frac{1}{\pi}\int \{\mathrm{d}^2\beta_k\} \prod_k \exp\{s|\beta_k|^2 + (\beta_k^*(\alpha_k - a_k) - \text{h.c.})\} \tag{5.1}$$

where $\{\mathrm{d}^2\beta_k\} = \prod_k \mathrm{d}^2\beta_k$. Then the s-ordered quasi-probability is determined by the expression

$$P(\{\alpha_k\}; s) = Sp\,(\Delta(\{\alpha_k\}, s)\rho). \tag{5.2}$$

Using the replacements $a_k \to a_m(l)$ in the above formulae we obtain the local quasi-probability $P(\{\alpha_m(l)\}; s)$. In the following we will restrict ourselves to the case $s = 1$, corresponding to the normal ordering of the field operators which is described by the Glauber–Sudarshan quasi-probability $P(s = 1) = P$. This function arises in the density matrix expansion over coherent states or the diagonal representation. From the given expressions it is possible to obtain the relation between local and non-local quasi-probabilities. Similarly in the case of operators, it is possible to proceed from one distribution function to another by the change of variables:

$$P(\{\alpha_m(l)\}) \quad \longleftrightarrow \quad P(\{\alpha_k\}) \tag{5.3}$$

$$\{\alpha_m(l)\} \quad \longleftrightarrow \quad \{\alpha_k\} \tag{5.4}$$

where the variables are related by the expression of (2.6) and (2.7).

In the Schrödinger representation the peculiarities of the local description specified by the many-particle character of the problem become apparent to the same extent as in the Heisenberg

representation. Here the transfer equations occur, which are formally equivalent to the BBGKI chains for the partial distribution functions. Out of all of the hierarchy of distribution functions in the present case, the following two types of one-particle quasi-probabilities are of interest. They arise from the function $P(\{\alpha_m(l)\})$, of all modes of all local oscillators. Averaging over all oscillators except for the chosen one we will find the distribution function $P(\alpha_{\{m\}}(l))$ for all modes of one local oscillator located at a point l. The averaging of $P(\alpha_{\{m\}}(l))$ over all modes except for one results in a function $P(\alpha_m(l))$ describing one mode of one local oscillator.

For a free evolution the introduced one-particle distribution functions $P_1 = P(\alpha_{\{m\}}(l))$, $P(\alpha_m(l))$ obey the following transfer equation:

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial l}\right) P_1 = 0. \tag{5.5}$$

In (5.5) the derivative over l describes the interaction of local oscillators or excitation transfer. Similarly to (2.13) it may be presented in the form

$$c\frac{\partial}{\partial l} P(\alpha_m(l)) = -h_m(l,l)P(\alpha_m(l)) - \sum_{l' \neq l} \int \mathrm{d}^2\alpha_m(l')\, h_m(l,l')P(\alpha_m(l), \alpha_m(l')) \tag{5.6}$$

where $h_m(l,l') = \mathrm{i}\Omega_m(l,l')(\partial/\partial\alpha_m(l))\alpha_m(l') + \text{c.c.}$ is the free evolution differential operator with the Hamiltonian $H_0 \leftrightarrow \sum_{mll'} h_m(l,l')$.

6. Fokker–Planck and Langevin equations

An approach based on the master equation for the quasi-probability $P(\{\alpha_k\})$ in the Fokker–Planck approximation is often used to describe the statistical properties of light. In the problem of light interaction with atoms the master equation for the electromagnetic field can be obtained by an adiabatic elimination of atomic variables. It is not necessary to derive the field equation once again in terms of the local description, because one may immediately use the change of variables (5.4).

As an example, consider the interaction of light with a two-level system in the lowest approximation for which the processes of linear amplification or absorption occur. Such a medium is described by a linear susceptibility, $\kappa(\omega) = |d|^2(\omega_0 - \omega - \mathrm{i}\gamma)^{-1}$ where d and ω_0 are the dipole moment and the frequency of the atomic transition, respectively, γ is the decay rate or transverse relaxation. The form of the field equation obtained by adiabatic elimination of the fast atomic variables is well known (see, [9]):

$$\frac{\partial}{\partial t} P(\{\alpha_k\}) = \sum_k \left(A(k)\frac{\partial}{\partial\alpha_k}\alpha_k + Q(k)\frac{\partial^2}{\partial\alpha_k\partial\alpha_k^*} + \text{c.c.}\right) P(\{\alpha_k\}). \tag{6.1}$$

Here the coefficients are defined by the linear susceptibility and populations of the upper N_2 and lower N_1 levels:

$$A(k) = \epsilon_k(N_1 - N_2)\,\mathrm{Im}\,\kappa(\omega_k) - \mathrm{i}\epsilon_k(N_1 + N_2)\,\mathrm{Re}\,\kappa(\omega_k)$$

$$Q(k) = \epsilon_k N_2\,\mathrm{Im}\,\kappa(\omega_k)$$

$$\epsilon_k = \hbar^{-2}(\hbar\omega_k/2\varepsilon_0 L^3).$$

When changing the variables we will assume that the wavepacket forming the local mode interacts as a whole. Then inside the band $k \sim m$ it is possible to neglect the dispersion of all harmonics. This means that $A(k) \approx A(m)$, $Q(k) \approx Q(m)$. This approximation enables us to

write at once the transfer equation for the one-particle quasi-probability, say $P(\alpha_{\{m\}}(l)) = P_1$ in the form

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial l}\right) P_1 = \sum_m \left(A(m)\frac{\partial}{\partial \alpha_m(l)}\alpha_m(l) + Q(m)\frac{\partial^2}{\partial \alpha_m(l)\partial \alpha_m^*(l)}\right) P_1 + \text{c.c.} \tag{6.2}$$

Note that the structure of the differential operators on the right-hand site is the same in (6.2) and (6.1), and the whole of the procedure of passing to the local description reduces to an addition of a derivative over l to the left-hand side of the equation.

The equation for the quasi-probability (6.1) is in agreement with the following Langevin equations:

$$\begin{aligned} \frac{\partial}{\partial t}\alpha_k &= -A(k)\alpha_k + f_k \\ \frac{\partial}{\partial t}\alpha_k^* &= -A^*(k)\alpha_k^* + f_k^\dagger \end{aligned} \tag{6.3}$$

where the correlator of random forces is defined by a diffusion coefficient

$$\langle f_{k'}(t) f_k^\dagger(t+\tau)\rangle = 2Q(k)\delta_{kk'}\delta(\tau). \tag{6.4}$$

In as much as the Langevin variables here are associated with the diagonal representation, they correspond to the normally ordered averages of the field operators $\langle a_k^\dagger(t)\, a_k(t+\tau)\rangle = \langle \alpha_k^*(t)\, \alpha_k(t+\tau)\rangle$.

In the local description, the main task in the formulation of the Langevin equations is in finding the random force correlator. To this end we use a change of variables of the form (2.19) and (2.20), where $A_m(l) \to \alpha_m(l)$, $a_k \to \alpha_k$, thereby we introduce the local Langevin variable $\alpha_m(l)$. Then the approximation $A(k) \approx A(m)$, $Q(k) \approx Q(m)$ used in writing down the equation for local quasi-probability leads to the Langevin equations

$$\begin{aligned} \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial l}\right)\alpha_m(l,t) &= -A(m)\,\alpha_m(l,t) + f_m(l) \\ \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial l}\right)\alpha_m^*(l,t) &= -A^*(m)\,\alpha_m^*(l,t) + f_m^\dagger(l). \end{aligned} \tag{6.5}$$

For the variables there follows an expression for a random source:

$$f_m(l,t) = \frac{1}{\sqrt{N}}\sum_{k\sim m} f_k \exp(-\mathrm{i}(\omega_k - \omega_m)t + \mathrm{i}(k-m)l). \tag{6.6}$$

From this we will find that the correlator of random sources, as well as in (6.4), is defined by the diffusion coefficient

$$\langle f_{m'}(l',t) f_m^\dagger(l,t+\tau)\rangle = 2Q(m)\delta_{mm'}\delta_{ll'}\delta(\tau). \tag{6.7}$$

As to (6.7), it should be noted that during the transition to a local description with a coarse space scale a scaling time δ-function should be used defined as in (2.17) and the l magnitude should be considered as a continuous coordinate. This implies that in (6.7) the following replacements take place: $\delta(\tau) \to \delta_a(\tau)$, $\delta_{ll'} \to a\delta(l-l')$. As a result the correlator of random sources in local Langevin equations has the form

$$\langle f_{m'l'}(t) f_{ml}^\dagger(t+\tau)\rangle = 2Q(m)\delta_{mm'}a\delta(l-l')\delta_a(\tau). \tag{6.8}$$

In comparison with the ordinary description in (6.8), a space delta-function arises due to the local correlation of oscillators in space.

7. Diffraction in a linear medium

The differential operator in (3.3) becomes non-Hermitian due to diffraction effects, so the corresponding equation for quasi-probability becomes very complicated. However, it is possible to use the Langevin formulation. In this way for the case of a linear medium considered above the change of variables in non-local Langevin equations (6.5) leads to Langevin equations, which in the three-dimensional case have the form

$$\begin{aligned}\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial z}-\mathrm{i}\frac{c^2}{2\omega_m}\left(\frac{\partial^2}{\partial x}+\frac{\partial^2}{\partial y}\right)\right)\alpha_m &= -A(m)\,\alpha_m + f_m(\boldsymbol{r},t)\\ \left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial z}+\mathrm{i}\frac{c^2}{2\omega_m}\left(\frac{\partial^2}{\partial x}+\frac{\partial^2}{\partial y}\right)\right)\alpha_m^* &= -A^*(m)\,\alpha_m^* + f_m^\dagger(\boldsymbol{r},t)\end{aligned} \tag{7.1}$$

where the random force correlator is defined as

$$\langle f_{m'}(\boldsymbol{r}',t)\,f_m^\dagger(\boldsymbol{r},t+\tau)\rangle = 2Q(m)\delta_{mm'}a^3\delta(\boldsymbol{r}-\boldsymbol{r}')\delta_a(\tau). \tag{7.2}$$

Here $Q(m)$ is the diffusion coefficient, which may be obtained from the corresponding equation (7.1), but non-local Fokker–Planck equation (6.1): $Q(m)\approx Q(k)$.

Equations (7.1) are linear in the field amplitudes, so they are easily integrated. Let us denote the transverse radius vector $\boldsymbol{s}=(x,y)$ and introduce a travelling coordinate system $t'=t-z/c$, $\mathrm{d}z'=z$. Then for the given conditions at the boundary one finds

$$\alpha_m(z,\boldsymbol{s},t)=\exp\left(-\frac{A}{c}z\right)\int \mathrm{d}^2\boldsymbol{s}_1\;\alpha_m(0,\boldsymbol{s}_1,t-z/c)\;U(0\boldsymbol{s}_1|z\boldsymbol{s})+W(z,\boldsymbol{s},t-z/c) \tag{7.3}$$

$$W=\frac{1}{c}\int_0^z \mathrm{d}z_1\exp\left(-\frac{A}{c}(z-z_1)\right)\int \mathrm{d}^2\boldsymbol{s}_1\;f_m(z_1,\boldsymbol{s}_1,t-z/c)U(z_1\boldsymbol{s}_1|z\boldsymbol{s}) \tag{7.4}$$

where the Green function is defined as

$$U(z_1\boldsymbol{s}_1|z\boldsymbol{s})=-\mathrm{i}\frac{m}{2\pi(z-z_1)}\exp\left(\mathrm{i}m\frac{(\boldsymbol{s}-\boldsymbol{s}_1)^2}{z-z_1}\right). \tag{7.5}$$

Here $\alpha_m(0,\boldsymbol{s}_1,t-z/c)$ is the transverse field distribution in the input, $A=A(m)$.

Consider the properties of a random source W, defining its correlation function

$$\langle W(z,0,t)W^\dagger(z,\boldsymbol{s},t+\tau)\rangle=\frac{2}{c^2}Qa^3\delta_a(\tau)\int_0^z\exp\left(-\frac{A+A^*}{c}(z-z_1)\right)D(z-z_1)\,\mathrm{d}z_1 \tag{7.6}$$

where the value

$$D=\left[\frac{m}{2\pi(z-z_1)}\right]^2\exp\left(-\mathrm{i}\frac{ms^2}{2(z-z_1)}\right)\int\mathrm{d}^2\boldsymbol{s}_1\exp\left(\mathrm{i}\frac{m}{z-z_1}\boldsymbol{s}_1\boldsymbol{s}\right) \tag{7.7}$$

describes the diffraction effects. Since the diffusion coefficient Q is determined by the number of atoms at the upper level N_2, the transverse sizes of the medium where the light propagates should be taken into account. In this way the limits of integration over $\boldsymbol{s}_1$ in D are defined. Thus $D=\delta(\boldsymbol{s})$ if the medium is not limited in the transverse direction. This means that the noise source is δ-correlated and the noise is white. Taking into consideration the finiteness of the size in the transverse direction changes the situation. Thus in a medium confined by a cylinder with a radius R

$$D=\frac{mR}{2\pi(z-z_1)s}J_1\left(\frac{msR}{z-z_1}\right)\exp\left(-\mathrm{i}\frac{ms^2}{2(z-z_1)}\right) \tag{7.8}$$

where J_1 is the Bessel function.

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